Miscellaneous Examples

Example 26 Write all the unit vectors in XY plane.

Solution Let $\vec{r} = x\hat{i} + y\hat{j}$ be a unit vector in XY-plane (Fig 10.28). Then, from the figure, we have $x = \cos\theta$ and $y = \sin\theta$ (since $|\vec{r}| = 1$). So, we may write the vector $\vec{r}$ as

$$\vec{r}\,(=\vec{OP}) = \cos\theta\,\hat{i} + \sin\theta\,\hat{j} \quad \ldots (1)$$

Clearly,

$$|\vec{r}| = \sqrt{\cos^2\theta + \sin^2\theta} = 1$$

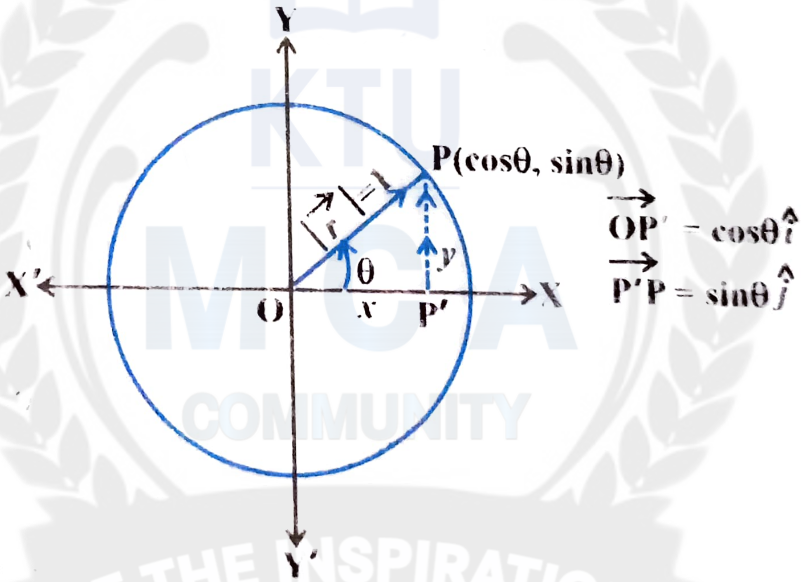

Fig 10.28

Also, as θ varies from 0 to 2π, the point P (Fig 10.28) traces the circle $x^2 + y^2 = 1$ counterclockwise, and this covers all possible directions. So, (1) gives every unit vector in the XY-plane.

Example 27 If $\hat{i}+\hat{j}+\hat{k}$, $2\hat{i}+5\hat{j}$, $3\hat{i}+2\hat{j}-3\hat{k}$ and $\hat{i}-6\hat{j}-\hat{k}$ are the position vectors of points A, B, C and D respectively, then find the angle between $\overrightarrow{AB}$ and $\overrightarrow{CD}$. Deduce that $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are collinear.

Solution Note that if θ is the angle between AB and CD, then θ is also the angle between $\overrightarrow{AB}$ and $\overrightarrow{CD}$.

Now $\overrightarrow{AB}$ = Position vector of B – Position vector of A

$$= (2\hat{i}+5\hat{j})-(\hat{i}+\hat{j}+\hat{k}) = \hat{i}+4\hat{j}-\hat{k}$$

Therefore $$|\overrightarrow{AB}| = \sqrt{(1)^2+(4)^2+(-1)^2} = 3\sqrt{2}$$

Similarly $$\overrightarrow{CD} = -2\hat{i}-8\hat{j}+2\hat{k} \text{ and } |\overrightarrow{CD}| = 6\sqrt{2}$$

Thus $$\cos\theta = \frac{\overrightarrow{AB}\cdot\overrightarrow{CD}}{|\overrightarrow{AB}||\overrightarrow{CD}|}$$

$$= \frac{1(-2)+4(-8)+(-1)(2)}{(3\sqrt{2})(6\sqrt{2})} = \frac{-36}{36} = -1$$

Since $0 \le \theta \le \pi$, it follows that $\theta = \pi$. This shows that $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are collinear.

Alternatively, $\overrightarrow{AB} = -\frac{1}{2}\overrightarrow{CD}$ which implies that $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are collinear vectors.

Example 28 Let $\vec{a}, \vec{b}$ and $\vec{c}$ be three vectors such that $|\vec{a}| = 3$, $|\vec{b}| = 4$, $|\vec{c}| = 5$ and each one of them being perpendicular to the sum of the other two, find $|\vec{a}+\vec{b}+\vec{c}|$.

Solution Given $\vec{a}\cdot(\vec{b}+\vec{c}) = 0$, $\vec{b}\cdot(\vec{c}+\vec{a}) = 0$, $\vec{c}\cdot(\vec{a}+\vec{b}) = 0$.

Now
$$|\vec{a}+\vec{b}+\vec{c}|^2 = (\vec{a}+\vec{b}+\vec{c})^2 = (\vec{a}+\vec{b}+\vec{c})\cdot(\vec{a}+\vec{b}+\vec{c})$$

$$= \vec{a}\cdot\vec{a}+\vec{a}\cdot(\vec{b}+\vec{c})+\vec{b}\cdot\vec{b}+\vec{b}\cdot(\vec{a}+\vec{c})$$

$$+ \vec{c}.(\vec{a}+\vec{b})+\vec{c}.\vec{c}$$

$$= |\vec{a}|^2+|\vec{b}|^2+|\vec{c}|^2$$

$$= 9+16+25 = 50$$

Therefore $$|\vec{a}+\vec{b}+\vec{c}| = \sqrt{50} = 5\sqrt{2}$$

Example 29 Three vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ satisfy the condition $\vec{a}+\vec{b}+\vec{c}=\vec{0}$. Evaluate the quantity $\mu = \vec{a}\cdot\vec{b} + \vec{b}\cdot\vec{c} + \vec{c}\cdot\vec{a}$, if $|\vec{a}|=1$, $|\vec{b}|=4$ and $|\vec{c}|=2$.

Solution Since $\vec{a}+\vec{b}+\vec{c}=\vec{0}$, we have

$$\vec{a}\cdot(\vec{a}+\vec{b}+\vec{c}) = 0$$

or

$$\vec{a}\cdot\vec{a}+\vec{a}\cdot\vec{b}+\vec{a}\cdot\vec{c} = 0$$

Therefore

$$\vec{a}\cdot\vec{b}+\vec{a}\cdot\vec{c} = -|\vec{a}|^2 = -1 \qquad \ldots (1)$$

Again,

$$\vec{b}\cdot(\vec{a}+\vec{b}+\vec{c}) = 0$$

or

$$\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c} = -|\vec{b}|^2 = -16 \qquad \ldots (2)$$

Similarly

$$\vec{a}\cdot\vec{c}+\vec{b}\cdot\vec{c} = -4. \qquad \ldots (3)$$

Adding (1), (2) and (3), we have

$$2\,(\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{a}\cdot\vec{c}) = -21$$

or

$$2\mu = -21, \text{ i.e., } \mu = \frac{-21}{2}$$

Example 30 If with reference to the right handed system of mutually perpendicular unit vectors $\hat{i}$, $\hat{j}$ and $\hat{k}$, $\vec{\alpha} = 3\hat{i} - \hat{j}$, $\vec{\beta} = 2\hat{i} + \hat{j} - 3\hat{k}$, then express $\vec{\beta}$ in the form $\vec{\beta} = \vec{\beta}_1 + \vec{\beta}_2$, where $\vec{\beta}_1$ is parallel to $\vec{\alpha}$ and $\vec{\beta}_2$ is perpendicular to $\vec{\alpha}$.

Solution Let $\vec{\beta}_1 = \lambda\vec{\alpha}$, λ is a scalar, i.e., $\vec{\beta}_1 = 3\lambda\hat{i} - \lambda\hat{j}$.

Now

$$\vec{\beta}_2 = \vec{\beta} - \vec{\beta}_1 = (2-3\lambda)\hat{i} + (1+\lambda)\hat{j} - 3\hat{k}.$$

Now, since $\vec{\beta}_2$ is to be perpendicular to $\vec{\alpha}$, we should have $\vec{\alpha}\cdot\vec{\beta}_2 = 0$. i.e.,

$$3(2-3\lambda) - (1+\lambda) = 0$$

or

$$\lambda = \frac{1}{2}$$

Therefore

$$\vec{\beta}_1 = \frac{3}{2}\hat{i} - \frac{1}{2}\hat{j} \quad \text{and} \quad \vec{\beta}_2 = \frac{1}{2}\hat{i} + \frac{3}{2}\hat{j} - 3\hat{k}$$

Summary

- Position vector of a point P(x, y, z) is given as $\overrightarrow{OP}(=\vec{r}) = x\hat{i} + y\hat{j} + z\hat{k}$, and its magnitude by $\sqrt{x^2 + y^2 + z^2}$.
- The scalar components of a vector are its direction ratios, and represent its projections along the respective axes.
- The magnitude (r), direction ratios (a, b, c) and direction cosines (l, m, n) of any vector are related as:

$$l = \frac{a}{r},\quad m = \frac{b}{r},\quad n = \frac{c}{r}$$

- The vector sum of the three sides of a triangle taken in order is $\vec{0}$.

- The vector sum of two coinitial vectors is given by the diagonal of the parallelogram whose adjacent sides are the given vectors.
- The multiplication of a given vector by a scalar λ, changes the magnitude of the vector by the multiple $|\lambda|$, and keeps the direction same (or makes it opposite) according as the value of λ is positive (or negative).
- For a given vector $\vec{a}$, the vector $\hat{a} = \frac{\vec{a}}{|\vec{a}|}$ gives the unit vector in the direction of $\vec{a}$.
- The position vector of a point R dividing a line segment joining the points P and Q whose position vectors are $\vec{a}$ and $\vec{b}$ respectively, in the ratio $m:n$

  (i) internally, is given by $\frac{n\vec{a}+m\vec{b}}{m+n}$.

  (ii) externally, is given by $\frac{m\vec{b}-n\vec{a}}{m-n}$.
- The scalar product of two given vectors $\vec{a}$ and $\vec{b}$ having angle θ between them is defined as

$$\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta .$$

  Also, when $\vec{a}\cdot\vec{b}$ is given, the angle 'θ' between the vectors $\vec{a}$ and $\vec{b}$ may be determined by

$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}$$

- If θ is the angle between two vectors $\vec{a}$ and $\vec{b}$, then their cross product is given as

$$\vec{a}\times\vec{b} = |\vec{a}||\vec{b}|\sin\theta\,\hat{n}$$

  where $\hat{n}$ is a unit vector perpendicular to the plane containing $\vec{a}$ and $\vec{b}$. Such that $\vec{a}, \vec{b}, \hat{n}$ form right handed system of coordinate axes.
- If we have two vectors $\vec{a}$ and $\vec{b}$, given in component form as $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ and λ any scalar,

then $\quad \vec{a}+\vec{b} = (a_1+b_1)\hat{i}+(a_2+b_2)\hat{j}+(a_3+b_3)\hat{k}$;

$$\lambda\vec{a} = (\lambda a_1)\hat{i}+(\lambda a_2)\hat{j}+(\lambda a_3)\hat{k};$$

$$\vec{a}.\vec{b} = a_1b_1+a_2b_2+a_3b_3;$$

and $\quad \vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}.$